

Network Synthesis for a Class of Mixed Quantum-Classical Linear Stochastic Systems

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Abstract

The purpose of this paper is to formulate and solve a synthesis problem for a class of linear quantum equations that may describe mixed quantum-classical systems. A *general* model and a *standard* model for mixed quantum-classical linear stochastic systems are proposed for our design processes and then we show how the former may be transformed into the latter, which can clearly present the internal structure of a mixed quantum-classical system. Physical realizability conditions are derived separately for the two models to ensure that they can correspond to physical systems. Furthermore, a network synthesis theory is developed for a mixed quantum-classical system of the standard form and an example is given to illustrate the theory.

Index Terms— linear stochastic system, mixed quantum-classical linear stochastic system, quantum system, network synthesis theory, physical realizability condition.

I. INTRODUCTION

Quantum technologies often comprise quantum systems interconnected with classical (non-quantum) devices. For instance, in quantum optics, an optical cavity may be part of a mixed quantum-classical system involving photodetectors, electronic amplifiers, piezoelectric actuators,

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feedback loops, etc [1], [2], [3]. Figure 1 illustrates an example of a mixed quantum-classical system, where two Fabry-Perot optical cavities [4], [5], [6] are connected to a classical controller via a homodyne detector (HD) and an electro-optic modulator (MOD), respectively [7], [8]. The classical controller processes the outcomes of a measurement of an observable of the cavity on the left hand side (e.g. the quadrature of an optical field). Modulating the quantum field with the classical controller output by MOD generates another quantum field sent to the cavity on the right. The signals from the classical controller also govern the behavior of the classical system, which can be implemented by electrical and electronic devices. Traditionally, such quantum optical networks would be implemented on an optics table. However, it is now becoming possible to consider implementation in semiconductor chips, [9], [10], [11]. In classical engineering,

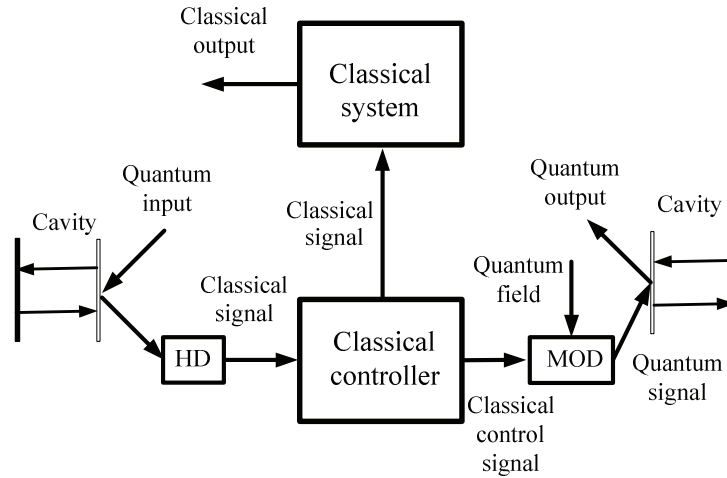


Fig. 1. A mixed quantum-classical system.

many methods have been developed for designing controllers and electronic systems. The design process begins with some form of specification for the system, and concludes with a physical realization of the system that meets the specifications. Often, mathematical models for the system are used in the design process, such as state space equations for the system. These state space equations may result from a mathematical optimization procedure, such as LQG, or some other procedure [12], [13], [14], [15]. The process of going from such mathematical models to the desired physical systems is a process of *synthesis* or *physical realization*, part of the design methodologies widely used in classical engineering [16]. The nature of the physical components to be used may restrict the range of, say, the state space models that can be used. For instance,

capacitors, inductors and resistors cannot by themselves implement non-passive devices like amplifiers.

Analogous design issues are beginning to present themselves in quantum technology. For example, linear quantum optics has been proposed as a means of implementing quantum information systems, [17]. Linear quantum optical systems may be described by linear quantum differential equations in the Heisenberg picture of quantum mechanics, [8], [11], [18], [19]. These equations look superficially like the classical state space equations familiar to engineering, but in fact are fundamentally different because they are equations for quantum mechanical operators, not numerical variables. The purpose of this paper is to consider synthesis problems for a class of linear stochastic differential equations that may describe mixed quantum-classical systems. This class of equations is usually presented in a general form given in Subsection III-A where the quantum-classical nature is captured in the matrices specifying the commutation relations of the system and signal (e.g. boson field) variables. However, the structure of a mixed quantum-classical system is not very clearly presented in a general form and we thus show how a mixed system described in general form can be linearly transformed into a standard form defined in Subsection III-B, which reveals in a standard (or canonical) way the internal structure of a mixed quantum-classical system. Furthermore, arbitrary linear stochastic differential equations for a general form or a standard form need not correspond to a physical system, and so we derive conditions ensuring that they do; that is, physical realizability. This work generalizes and extends earlier work [20], [21], [22]. In [22], we only consider a *standard* model for mixed quantum-classical linear stochastic systems for the design process. However, in this paper, we will investigate a more general model for the physical realization of the mixed quantum-classical linear stochastic system.

This paper is organized as follows. Section II introduces some notations and gives a brief overview of quantum and mixed quantum-classical linear stochastic systems as well as quantum non-demolition measurement and non-demolition conditions. Section III proposes two models of mixed quantum-classical linear stochastic systems for the design process and presents a connection between these models. Section IV presents physical realizability definitions and constraints for the two models defined in Section III, respectively. Section V develops a network synthesis theory for a mixed quantum-classical system of the standard form, followed by one example. Finally, Section VI gives the conclusion of this paper.

II. PRELIMINARIES

A. Notation

The notations used in this paper are as follows: $i = \sqrt{-1}$; the commutator is defined by $[A, B] = AB - BA$. If x and y are column vectors of operators, the commutator is defined by $[x, y^T] = xy^T - (yx^T)^T$. If $X = [x_{jk}]$ is a matrix of linear operators or complex numbers, then $X^\# = [x_{jk}^*]$ denotes the operation of taking the adjoint of each element of X , and $X^\dagger = [x_{jk}^*]^T$. We also define $\Re(A) = (X + X^\#)/2$ and $\Im(X) = (X - X^\#)/2i$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $\text{diag}_n(M)$ denotes a block diagonal matrix with a square matrix M appearing n times on the diagonal block. The symbol I_n denotes the $n \times n$ identity matrix. $0_{n \times m}$ denotes the $n \times m$ zero matrix, where n and m can be determined from context when the subscript is omitted.

B. Quantum Linear Stochastic Systems and Physical Realizability

Consider an *open quantum harmonic oscillator* ([20, Theorem 3.4]) consisting of n one degree of freedom open quantum harmonic oscillators coupled to boson fields (e.g. optical beams), [8], [18]. Each oscillator may be represented by position q_j and momentum p_j operators ($j = 1, \dots, n$), while each field channel is described by analogous field operators $w_{q_k}(t)$, $w_{p_k}(t)$, ($k = 1, \dots, m$). The oscillator variables are *canonical* if they satisfy the canonical commutation relations $[q_j, p_k] = 2i\delta_{jk}$ ($j, k = 1, \dots, n$). In vector form, we write $\xi = [q_1, p_1, q_2, p_2, \dots, q_n, p_n]^T$, and the commutation relations become

$$\xi\xi^T - (\xi\xi^T)^T = 2i\Theta, \quad (1)$$

where in the canonical case, $\Theta = \text{diag}_n(J)$. Similarly, the Ito products for the fields $w = [w_{q_1}, w_{p_1}, w_{q_2}, w_{p_2}, \dots, w_{q_m}, w_{p_m}]^T$ may be written as

$$dw(t)dw(t)^T = F_w dt, \quad (2)$$

where in the canonical case $F_w = I_{2m} + i\text{diag}_m(J)$. Commutation relations for the noise components of w can be defined as:

$$[dw(t), dw(t)^T] = (F_w - F_w^T)dt = 2i\Theta_w dt.$$

The dynamical evolution of an open system is unitary (in the Hilbert space consisting of the system and fields), and in the Heisenberg picture the system variables and output field operators evolve according to equations of the form

$$\begin{aligned} d\xi(t) &= \mathcal{A}\xi(t)dt + \mathcal{B}dw(t), \\ dz(t) &= \mathcal{C}\xi(t)dt + \mathcal{D}dw(t) \end{aligned} \quad (3)$$

with real constant matrices \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} satisfying

$$\mathcal{A}\Theta + \Theta\mathcal{A}^T + \mathcal{B}\Theta_w\mathcal{B}^T = 0, \quad (4)$$

$$\mathcal{B}\mathcal{D}^T = \Theta\mathcal{C}^T\Theta_w, \quad (5)$$

$$\mathcal{D} = I_{2m} \quad \text{or} \quad [I_{2n_z} \quad 0], \quad (6)$$

where $2n_z$ and $2m$ are the dimensions of the output z and input w , respectively. We see therefore that in the Heisenberg picture dynamical “state space” equations look formally like the familiar state space equations in classical systems and control theory. However, for arbitrary matrices \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} , equations (3) need not correspond to a canonical open oscillator. The system (3) is said to be *physically realizable* if the equations (3) correspond to an open quantum harmonic oscillator, [20, Definition 3.3]). As shown in [20], the system (3) with \mathcal{D} defined as in (6) is physically realizable if and only if the matrices \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} satisfy conditions (4) and (5). In general, we may take the commutation matrix Θ to be skew-symmetric, while the Ito matrix F is non-negative Hermitian. These generalizations, which will be used in Subsection III-A, allow us to consider classical variables, characterized by zero commutation relations, as well as classical noise processes, corresponding to the absence of the imaginary part in the Ito products, [20], [21], [23].

C. Mixed Quantum-Classical Linear Stochastic Systems with Quantum Inputs and Quantum Outputs

Now we let x have quantum and classical degrees of freedom, such that $x = [x_q^T, x_c^T]^T$, where classical variables $x_c(t)$ commute with one another and with the degrees of freedom in quantum variables $x_q(t)$. Thus, the commutation relation for $x(t)$ satisfies

$$xx^T - (xx^T)^T = 2i\Theta_n,$$

where $\Theta_n = \text{diag}(\Theta_{n_q}, 0_{n_c \times n_c})$ with $\Theta_{n_q} = \text{diag}_{n_q}(J)$ is said to be *degenerate canonical* by the terminology of [20].

Consider a mixed quantum-classical linear stochastic system in terms of x given by

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t), \\ dy_q(t) &= C_q x(t)dt + D_q dw(t), \end{aligned} \quad (7)$$

where w is defined in Subsection II-B; $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 2m}$, $C_q \in \mathbb{R}^{2n_{yq} \times n}$ and $D_q \in \mathbb{R}^{2n_{yq} \times 2m}$, $n = 2n_q + n_c$. If we are given a component of a vector of classical system variables x_c denoted by x_{c_k} , we may consider x_{c_k} as one of the quadratures of a quantum harmonic oscillator, say the position quadrature q_k . The vector $\tilde{x}_k(t) = \begin{bmatrix} q_k(t) \\ p_k(t) \end{bmatrix}$ is called an *augmentation* of $x_{c_k}(t)$. That is, $x(t)$ can be embedded in a larger vector $\tilde{x}(t) = [x(t)^T \quad \eta(t)^T]^T$, where any element of $\eta(t) = [\eta_1(t), \eta_2(t), \dots, \eta_{n_c}(t)]^T$ commute with any component of $x_q(t)$, and are conjugate to the components of $x_c(t)$, satisfying $[x_{c,j}(t), \eta_k(t)] = 2i\delta_{jk}$, where δ_{jk} is the Kronecker delta function. Then the commutation relation for $\tilde{x}(t)$ is defined as $\tilde{x}\tilde{x}^T - (\tilde{x}\tilde{x}^T)^T = 2i\tilde{\Theta}$. So, the *augmented* system of the system (7) in terms of \tilde{x} can be defined as:

$$d\tilde{x}(t) = \tilde{A}\tilde{x}(t)dt + \tilde{B}dw(t), \quad (8)$$

$$d\tilde{y}_q(t) = \tilde{C}\tilde{x}(t)dt + \tilde{D}dw(t), \quad (9)$$

where $\tilde{A} = \begin{bmatrix} A & 0 \\ A' & A'' \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ B' \end{bmatrix}$, $\tilde{C} = \begin{bmatrix} C_q & 0 \end{bmatrix}$, $\tilde{D} = D_q$, $\tilde{y}_q = y_q$, $\tilde{\Theta} = \begin{bmatrix} \Theta_n & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \begin{bmatrix} 0 & -I \end{bmatrix} & 0 \end{bmatrix}$

is a invertible matrix with $\tilde{\Theta}\tilde{\Theta} = -I$ and $\tilde{\Theta} = -\tilde{\Theta}^T$. The matrices A', A'', B' will be given in the proof of Theorem 3.

The system (7) is said to be physically realizable if its corresponding *augmented* system described by (8)-(9) can represent the dynamics of an open quantum harmonic oscillator after a suitable relabeling of the components of the variables $\tilde{x}(t)$. Recalling the results of [20], we then have the following theorem.

Theorem 1: A mixed quantum-classical system (7) with quantum inputs and quantum outputs is physically realizable, where $D_q = I_{2m}$ or $D_q = [I_{2n_{yq}} \quad 0]$, if and only if A, B, C_q satisfy the conditions (4) and (5) with matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and Θ replaced by corresponding matrices A, B, C_q, D_q and Θ_n , respectively.

D. Quantum Non-demolition Condition

The Belavkin's nondemolition principle requires an observable $X(t)$ at a time instant t to be compatible with the past output process $Y(s)$ ($s \leq t$) [24], [25], [26], that is:

$$[X(t), Y(s)^T] = 0, \quad \forall t \geq s \geq 0. \quad (10)$$

Condition (10) is known as *non-demolition condition*.

III. MIXED QUANTUM-CLASSICAL LINEAR STOCHASTIC MODELS

In this section, we will give two models or forms (a general form and a *standard* form defined later) for mixed quantum-classical linear stochastic systems and then derive relations between two models. We allow the general form to include classical inputs and outputs, which are not considered in previous works [20], [21].

A. A General Form for Mixed Linear Stochastic Systems with Mixed Inputs and Mixed Outputs

Consider a general form for linear mixed quantum-classical stochastic systems given by

$$\begin{aligned} d\mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t)dt + \mathbf{B}dv(t), \\ d\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)dt + \mathbf{D}dv(t), \end{aligned} \quad (11)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{n_y \times n}$ and $\mathbf{D} \in \mathbb{R}^{n_y \times m}$; $x(t)$ includes quantum and classical system variables satisfying the commutation relation, such that $\mathbf{x}_0\mathbf{x}_0^T - (\mathbf{x}_0\mathbf{x}_0^T)^T = 2i\mathbf{\Theta}_n$ with a skew-symmetric matrix $\mathbf{\Theta}_n$ ($\mathbf{x}(0) = \mathbf{x}_0$); the vector $v(t)$ represents the input signals, which contains quantum and classical noises; $\mathbf{y}(t)$ represents mixed quantum-classical outputs. F_v and F_y are nonnegative definite Hermitian matrices satisfying $dv(t)dv(t)^T = F_v dt$ and $dy(t)dy(t)^T = F_y dt$. The transfer function $\Xi_G(s)$ for a system of the form (11) is denoted by

$$\Xi_G(s) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right] (s) = \mathbf{C}(sI_n - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}.$$

B. A Standard Form for Mixed Linear Stochastic Systems with Quantum Inputs and Mixed Outputs

From the *general* form (11), it is not obvious to identify which parts are quantum components while which parts correspond to classical components. Therefore, we need to transform the system (11) into a form (called *standard* form), which presents a clear structure of a mixed quantum-classical system. Consider a *standard* form given by

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t), \\ dy(t) &= Cx(t)dt + Ddw(t), \end{aligned} \quad (12)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 2m}$, $C \in \mathbb{R}^{n_y \times n}$ and $D \in \mathbb{R}^{n_y \times 2m}$. $y = [y_q^T \ y_c^T]^T$ with $y_q \in \mathbb{R}^{2n_{yq}}$ and $y_c \in \mathbb{R}^{n_{yc}}$, $w = [w_1^T \ w_2^T]^T$ with $w_1 \in \mathbb{R}^{2n_{w1}}$ and $w_2 \in \mathbb{R}^{2n_{w2}}$. Here $m = n_{w1} + n_{w2}$, $n_y = n_{\mathbf{y}} = 2n_{yq} + n_{yc}$. Let initial values $x(0) = x_0$ satisfy the commutation relations: $x_0 x_0^T - (x_0 x_0^T)^T = 2i\Theta_n$. We assume that $\Theta_w = \frac{F_w - F_w^T}{2i}$ with $dw(t)dw(t)^T = F_w dt$ and $\Theta_y = \frac{F_y - F_y^T}{2i}$ with $dy(t)dy(t)^T = F_y dt$. The transfer function for the system of the form (12) is given by

$$\Xi_S(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] (s) = C(sI_n - A)^{-1}B + D.$$

Definition 1: A mixed quantum-classical linear stochastic system of the form (12) is said to be *standard* if the following statements are satisfied:

- 1) $\Theta_n = \text{diag}(\Theta_{n_q}, 0_{n_c \times n_c})$ with $\Theta_{n_q} = \text{diag}_{n_q}(J)$ and $2n_q + n_c = n$ ($n_c \geq 0$).
- 2) $\Theta_w = \text{diag}_m(J)$.
- 3) $F_y = I_{n_y} + \text{diag}(\Theta_{y_q}, 0_{n_{yc} \times n_{yc}})$, where $n_y = 2n_{yq} + n_{yc}$ ($n_{yq} \leq m$).

Let the matrices A and C be partitioned compatibly with partitioning of $x(t)$ into $x_q(t)$ and $x_c(t)$ as $A = \begin{bmatrix} A_{qq} & A_{qc} \\ A_{cq} & A_{cc} \end{bmatrix}$ and $C = \begin{bmatrix} C_q \\ C_c \end{bmatrix} = \begin{bmatrix} C_{qq} & C_{qc} \\ C_{cq} & C_{cc} \end{bmatrix}$. Let the matrices B and D

be partitioned according to partitioning of $w(t)$ into $w_1(t)$ and $w_2(t)$ as $B = \begin{bmatrix} B_q \\ B_c \end{bmatrix}$ and $D = \begin{bmatrix} D_q \\ D_c \end{bmatrix}$. Let $y(t)$ be partitioned into $y_q(t)$ and $y_c(t)$. Then, the system (12) can be

rewritten as follows:

$$dx_q(t) = \begin{bmatrix} A_{qq} & A_{qc} \end{bmatrix} x(t)dt + B_q dw(t), \quad (13)$$

$$dx_c(t) = \begin{bmatrix} A_{cq} & A_{cc} \end{bmatrix} x(t)dt + B_c dw(t), \quad (14)$$

$$dy_q(t) = \begin{bmatrix} C_{qq} & C_{qc} \end{bmatrix} x(t)dt + D_q dw(t), \quad (15)$$

$$dy_c(t) = \begin{bmatrix} C_{cq} & C_{cc} \end{bmatrix} x(t)dt + D_c dw(t), \quad (16)$$

where $A_{qq} \in \mathbb{R}^{2n_q \times 2n_q}$, $A_{qc} \in \mathbb{R}^{2n_q \times n_c}$, $A_{cq} \in \mathbb{R}^{n_c \times 2n_q}$, $A_{cc} \in \mathbb{R}^{n_c \times n_c}$, $B_q \in \mathbb{R}^{2n_q \times 2m}$, $B_c \in \mathbb{R}^{n_c \times 2m}$, $C_{qq} \in \mathbb{R}^{2n_{yq} \times 2n_q}$, $C_{qc} \in \mathbb{R}^{2n_{yq} \times n_c}$, $C_{cq} \in \mathbb{R}^{n_{yc} \times 2n_q}$, $C_{cc} \in \mathbb{R}^{n_{yc} \times n_c}$, $D_q \in \mathbb{R}^{2n_{yq} \times 2m}$, $D_c \in \mathbb{R}^{n_{yc} \times 2m}$.

Remark 1: The first item of Definition 1 indicates that $x(t)$ has both quantum and classical degrees of freedom, where Θ_{n_q} corresponds to the quantum degrees of freedom x_q while $0_{n_c \times n_c}$ corresponds to the classical degrees of freedom x_c . The second item of Definition 1 shows that input signals of the system (12) must be fully quantum. The third item of Definition 1 implies that

$$\Theta_y = D\Theta_w D^T = \text{diag}(\Theta_{y_q}, 0_{n_{yc} \times n_{yc}}), \quad (17)$$

where $\Theta_{y_q} = \text{diag}_{n_{yq}}(J)$ corresponds to quantum outputs while the matrix $0_{n_{yc} \times n_{yc}}$ corresponds to classical outputs, which will be discussed further and proved under suitable hypotheses in Section IV. So, the difference between the mixed linear systems (7) and (12) is that the latter explicitly exhibits classical output signals, and the matrix D has a more general form satisfying condition (17), which is equivalent to the following equations:

$$D_q \Theta_w D_q^T = \Theta_{y_q}, \quad (18)$$

$$D_q \Theta_w D_c^T = 0, \quad (19)$$

$$D_c \Theta_w D_c^T = 0. \quad (20)$$

C. Relations between the General and Standard Forms

The general form (11) and the *standard* form (12) can be related by the following lemmas and theorem:

Lemma 1: Given an arbitrary $n \times n$ real skew-symmetric matrix Θ_n ($n \geq 2$), there exists a real nonsingular matrix P_n and a block diagonal matrix $\Theta_n = \text{diag}(\Theta_{n_q}, 0_{n_c \times n_c})$ such that

$$\Theta_n = P_n \Theta_n P_n^T. \quad (21)$$

The similar proof of Lemma 1 can be found in [27] and hence is omitted here.

Lemma 2: Given an arbitrary $m \times m$ nonnegative definite Hermitian matrix F_v , there exists a $2m \times 2m$ matrix $F_w = I_{2m} + i \text{diag}_m(J)$ and a $m \times 2m$ real matrix W such that

$$F_v = W F_w W^T. \quad (22)$$

Proof: Hermitian matrices F_v and F_w can be diagonalized by unitary matrices U_v and U_w , respectively, such that

$$F_v = U_v \Lambda_v U_v^\dagger, \quad (23)$$

$$F_w = U_w \Lambda_w U_w^\dagger, \quad (24)$$

where $\Lambda_v = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$, ($\lambda_j \geq 0$ is an eigenvector of F_v), $\Lambda_w = \text{diag}_m \left(\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right)$,

$U_w = \text{diag}_m \left(\frac{\sqrt{2}}{2} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \right)$. Since Λ_v and Λ_w are two real diagonal matrices, there exists a $m \times 2m$ complex matrix $Q = [q_1, q_2, \dots, q_{2m}]$ such that

$$\Lambda_v = Q \Lambda_w Q^\dagger. \quad (25)$$

In order to let (25) hold, for simplicity we choose $q_2 = \left[\sqrt{\frac{\lambda_1}{2}} 0 \dots 0 \right]^T$, $q_4 = \left[0 \sqrt{\frac{\lambda_2}{2}} 0 \dots 0 \right]^T$, \dots , $q_{2m} = \left[0 \dots 0 \sqrt{\frac{\lambda_m}{2}} \right]^T$, and $q_1, q_3, \dots, q_{2m-1}$ now are arbitrary column vectors of length m and to be determined later. Combining (23), (24) and (25) gives

$$F_v = U_v Q U_w^\dagger F_w (U Q U_w^\dagger)^\dagger. \quad (26)$$

Let W be defined as $W = U_v Q U_w^\dagger$. Then we have

$$U_v Q = [U_v q_1, U_v q_2, \dots, U_v q_{2m}]. \quad (27)$$

Next, we will show that Q can be chosen to let W be real. Observing the structure of U_w , such that

$$U_w = \text{diag}_m \left(\frac{\sqrt{2}}{2} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \right),$$

we require that $q_1, q_3, \dots, q_{2m-1}$ be chosen as

$$\begin{aligned} q_1 &= -U_v^\dagger U_v^\# q_2, \\ q_3 &= -U_v^\dagger U_v^\# q_4, \\ &\vdots \\ q_{2m-1} &= -U_v^\dagger U_v^\# q_{2m}. \end{aligned}$$

The matrix Q is hence constructed as

$$Q = \begin{bmatrix} -U_v^\dagger U_v^\# q_2, & q_2, & -U_v^\dagger U_v^\# q_4, & q_4, & \dots & -U_v^\dagger U_v^\# q_{2m}, & q_{2m} \end{bmatrix}.$$

We can get the representation (22) with $W = U_v Q U_w^\dagger$. ■

Let us look at an example applying Lemma 2.

Example 1 : Consider a nonnegative definite Hermitian matrix given by

$$F_v = \begin{bmatrix} 8.9286 & -0.2143 + 4.8107i & 0.1429 + 7.2161i \\ -0.2143 - 4.8107i & 8.3571 + 0.0000i & 0.4286 - 2.4054i \\ 0.1429 - 7.2161i & 0.4286 + 2.4054i & 8.7143 \end{bmatrix}.$$

It is easily obtained that $F_v = U_v \Lambda_v U^\dagger$ with

$$U_v = \begin{bmatrix} 0.6814 & 0.6814 & 0.2673 \\ -0.1572 - 0.3922i & -0.1572 + 0.3922i & 0.8018 \\ 0.1048 - 0.5883i & 0.1048 + 0.5883i & -0.5345 \end{bmatrix}$$

and

$$\Lambda_v = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Now following the construction in the proof of Lemma 2, we want to find a real matrix

W . Choosing $q_2 = [3 \ 0 \ 0]^T$, $q_4 = [0 \ 0 \ 0]^T$ and $q_6 = [0 \ 0 \ 2]^T$ we get $q_1 = [0 \ -3 \ 0]^T$, $q_3 = [0 \ 0 \ 0]^T$ and $q_5 = [0 \ 0 \ -2]^T$. So the matrix $Q = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}$.

It follows from the above construction that $W = \begin{bmatrix} 0 & 2.8909 & 0 & 0 & 0 & 0.7559 \\ -1.6641 & -0.6671 & 0 & 0 & 0 & 2.2678 \\ -2.4962 & 0.4447 & 0 & 0 & 0 & -1.5119 \end{bmatrix}$.
It is easily checked that $F_v = W F_w W^T$ with $F_w = I_6 + i \text{diag}_3(J)$.

Theorem 2: Given a mixed quantum-classical stochastic system of the general form (11), there exists a corresponding *standard* form (12).

Proof: By Lemma 1 and 2, there exist matrices P_n , W and P_y , such that

$$\left. \begin{aligned} \Theta_n &= P_n \Theta_n P_n^T, & \Theta_v &= W \Theta_w W^T, & y &= P_y y \\ \text{diag}(\Theta_{y_q}, 0_{n_{y_c} \times n_{y_c}}) &= P_y \Theta_y P_y^T, & A &= P_n \mathbf{A} P_n^{-1} \\ B &= P_n \mathbf{B} W, & C &= P_y \mathbf{C} P_n^{-1}, & D &= P_y \mathbf{D} W \end{aligned} \right\}. \quad (28)$$

Substituting relations (28) into (11) gives (12). Now, we can verify the following relation between the *standard* $\Xi_S(s)$ and general $\Xi_G(s)$ transfer functions:

$$\begin{aligned} \Xi_S(s) &= C (sI_n - A)^{-1} B + D \\ &= P_y \mathbf{C} P_n^{-1} (sP_n P_n^{-1} - P_n \mathbf{A} P_n^{-1})^{-1} P_n \mathbf{B} W + P_y \mathbf{D} W \\ &= P_y (\mathbf{C} (sI_n - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}) W \\ &= P_y \Xi_G(s) W. \end{aligned}$$

Thus, the general form (11) can be linearly transformed into its corresponding *standard* form (12).

■

IV. PHYSICAL REALIZABILITY OF MIXED QUANTUM-CLASSICAL LINEAR STOCHASTIC SYSTEMS

In this section, we will introduce the definition of physical realizability of the *standard* form and a theorem on necessary and sufficient conditions for physical realizability. Analogous physical realizability definition and conditions for the general form are also presented in this section.

A. Physical Realizability for the Standard Form

The following lemmas will be used for introducing the definition of physical realizability of the system (12).

Lemma 3: Non-demolition condition $[\tilde{x}(t), y_q(s)^T] = 0, \forall t \geq s \geq 0$ for the *augmented* system (8)-(9) of the system (7) holds, if and only if

$$\tilde{B}\Theta_w\tilde{D}^T = -\tilde{\Theta}\tilde{C}^T. \quad (29)$$

Proof: First, we will argue that $[\tilde{x}(t), y_q(s)^T] = 0$ is equivalent to $[\tilde{x}(t), y_q^T(t)] = 0$, for all $t \geq s \geq 0$. Let $g_s(t) = [\tilde{x}(t), y_q(s)^T]$, for all $t \geq s \geq 0$, where s is fixed. From $[\tilde{x}(t), y_q(t)^T] = 0$ for all $t \geq s \geq 0$, we can infer that $g_s(s) = 0$ and then have

$$\begin{aligned} dg_s(t) &= d[\tilde{x}(t), y_q(s)^T] \\ &= [d\tilde{x}(t), y_q(s)^T] \\ &= \tilde{A}[\tilde{x}(t), y_q(s)^T]dt \\ &= \tilde{A}g_s(t)dt. \end{aligned}$$

Solving the above equation gives

$$g_s(t) = \exp\left(\tilde{A}(t-s)\right)g_s(s) = 0.$$

Therefore, $[\tilde{x}(t), y_q(t)^T] = 0$ implies $[\tilde{x}(t), y_q(s)^T] = 0$, for all $t \geq s \geq 0$. Conversely, it is trivial to verify that $[\tilde{x}(t), y_q(s)^T] = 0$ for all $t \geq s \geq 0$ implies $[\tilde{x}(t), y_q(t)^T] = 0$ for all $t \geq 0$.

Thus, we just need to consider the case where $t = s$. Let $g(t) = [\tilde{x}(t), y_q(t)^T]$ with $g(0) = 0$ and then we have

$$\begin{aligned} dg(t) &= d[\tilde{x}(t), y_q(t)^T] \\ &= [d\tilde{x}(t), y_q(t)^T] + [\tilde{x}(t), dy_q(t)^T] + [d\tilde{x}(t), dy_q(t)^T] \\ &= \tilde{A}g(t)dt + 2i(\tilde{\Theta}\tilde{C}^T + \tilde{B}\Theta_w\tilde{D}^T)dt. \end{aligned}$$

Solving the above equation gives

$$g(t) = \exp(\tilde{A}t)g(0) + 2i \int_0^t \exp(\tilde{A}(t-\tau)) \left(\tilde{\Theta}\tilde{C}^T + \tilde{B}\Theta_w\tilde{D}^T \right) d\tau. \quad (30)$$

It can be easily verified from (30) that $g(t) = 0$ holds for all $t \geq 0$, if and only if

$$\tilde{\Theta}\tilde{C}^T + \tilde{B}\Theta_w\tilde{D}^T = 0.$$

■

Lemma 4: Non-demolition condition $[x(t), y(s)^T] = 0, \forall t \geq s \geq 0$ for the system (12) holds, if and only if

$$B\Theta_w D^T = -\Theta_n C^T. \quad (31)$$

The proof of Lemma 4 is similar to that of Lemma 3 and is thus omitted.

For a better understanding of Definition 2 and 3 below, a discussion regarding the physical realizability of the *standard* form (12) will be given first. The system (12) can be divided into two parts: one is the system (7) with D_q satisfying (18), or equivalently described by (13)-(15); the other is the output equation (16). So, the system (12) is physically realizable if the two parts are both physically realizable. First, we consider physical realizability conditions of the system (7). From the structure of system matrices of the *augmented* system (8)-(9), it is clear that the dynamics of $x(t)$ of system (7) embedded in system (8)-(9) are not affected by the augmentation, and matrices A', A'', B' in system (8)-(9) can be chosen to preserve commutation relations for augmented system variables \tilde{x} shown in the proof of Theorem 3. Motivated by the results in [20], we want to argue that the system (7) with $\tilde{D} = D_q$ satisfying (18) is physically realizable if its *augmented* system (8)-(9) can be physically realizable. However, the previous definition and theorem of physical realizability in [20] are only suitable for an *augmented* system (8)-(9) with $\tilde{D} = I$ or $\tilde{D} = [I \ 0]$ (no scattering processes involved). We hence need to transform the *augmented* system (8)-(9) into a familiar form without scattering processes. Suppose that non-demolition condition $[\tilde{x}(t), y_q(s)^T] = 0, \forall t \geq s \geq 0$ holds. So, we apply relation (29) in Lemma 3 to the output (9) to get $y_q = \tilde{D}\bar{y}_q$ with \bar{y}_q defined as $d\bar{y}_q = \bar{C}\tilde{x}(t)dt + dw(t)$, where $\bar{C} = \Theta_w \tilde{B}^T \tilde{\Theta}$. Then, a *reduced* system for the *augmented* system (8)-(9) is defined as

$$\begin{aligned} d\tilde{x}(t) &= \tilde{A}\tilde{x}(t)dt + \tilde{B}dw(t), \\ d\bar{y}_q &= \bar{C}\tilde{x}(t)dt + dw(t). \end{aligned} \quad (32)$$

It is straightforward to verify that the *reduced* system (32) is physically realizable in the sense of Theorem 1. The definition of physical realizability of an *augmented* system of the system (7) is as follows:

Definition 2: An *augmented* system (8)-(9) of the form (7) is said to be physically realizable if the following statements hold:

- 1) The *reduced* system (32) is physically realizable in the sense of Theorem 1.

- 2) For the *augmented* system (8)-(9), non-demolition condition $[\tilde{x}(t), y_q(s)^T] = 0, \forall t \geq s \geq 0$ holds.
- 3) $\tilde{D} = D_q$ is of the form $[I_{n_{yq}}, 0]\tilde{V}$ with \tilde{V} a symplectic matrix [28] or unitary symplectic [4] such that relation (18) holds.

Next we will consider physical realizability conditions of the system (16). Classical systems are always regarded as being physically realizable since they can be approximately built via digital and analog circuits. Thus, we just need to make sure that output equation (16) is classical. Now, we can present a formal definition of physical realizability of the system (12).

Definition 3: A system of the *standard* form (12) is said to be physically realizable if the following statements hold:

- 1) There exists an *augmented* system (8)-(9) of the system (7) with D_q satisfying (18), which is physically realizable in the sense of Definition 2.
- 2) For the system (12), non-demolition condition $[x(t), y(s)^T] = 0, \forall t \geq s \geq 0$ holds.
- 3) The output (16) and system variables x_c both represent classical stochastic processes in the sense of commutation relations $[x_c(t), x_c^T(s)] = 0, [x_c(t), y_c^T(s)] = 0, [y_c(t), y_c^T(s)] = 0$, for all $t, s \geq 0$, where $[x_c(0), y_c(0)^T] = 0$ and $[y_c(0), y_c(0)^T] = 0$.

The following theorem shows necessary and sufficient conditions for physical realizability.

Theorem 3: A system of the form (12) is physically realizable, if and only if matrices A, B, C , and D satisfy the following constraints:

$$A\Theta_n + \Theta_n A^T + B\Theta_w B^T = 0, \quad (33)$$

$$B\Theta_w D^T = -\Theta_n C^T, \quad (34)$$

$$D\Theta_w D^T = \text{diag}(\Theta_{yq}, 0_{n_{yc} \times n_{yc}}). \quad (35)$$

Proof: First, let conditions (33)-(35) hold. ① Multiplying both sides of (34) by $\begin{bmatrix} I_{2n_{yq}} \\ 0 \end{bmatrix}$, we get

$$B\Theta_w D_q^T = -\Theta_n C_q^T. \quad (36)$$

It follows by inspection that under conditions (33) and (36), there exist matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ and $\tilde{\Theta}$ defined in Subsection II-C satisfying the following conditions

$$\tilde{A}\tilde{\Theta} + \tilde{\Theta}\tilde{A}^T + \tilde{B}\Theta_w\tilde{B}^T = 0, \quad (37)$$

$$\tilde{C} = \tilde{D}\Theta_w\tilde{B}^T\tilde{\Theta}, \quad (38)$$

$$\tilde{D} = D_q, \quad (39)$$

where A', A'', B' satisfy the following relations:

$$B'\Theta_w D_q^T = [0 \quad I]C_q^T, \quad (40)$$

$$\begin{bmatrix} 0 & I \end{bmatrix} A'^T - A' \begin{bmatrix} 0 \\ I \end{bmatrix} = B'\Theta_w B'^T, \quad (41)$$

$$A'' = (A'\Theta_n - [0 \quad I]A^T + B'\Theta_w B^T) \begin{bmatrix} 0 \\ I \end{bmatrix}. \quad (42)$$

From (9) and (38), we get

$$\bar{C} = \Theta_w \tilde{B}^T \tilde{\Theta}. \quad (43)$$

So, conditions (37) and (43) imply the *reduced* system (32) is physically realizable in the sense of Theorem 1. By Lemma 3, condition (38) implies that $[\tilde{x}(t), y_q(s)^T] = 0, \forall t \geq s \geq 0$ holds, which satisfies the second condition of Definition 2. Multiplying both sides of (35) by $[I \quad 0]$ and $\begin{bmatrix} I \\ 0 \end{bmatrix}$, we can obtain (18). Thus, the *augmented* system (8)-(9) is physically realizable in the sense of Definition 2. ② By Lemma 4, condition (34) implies that $[x(t), y(s)^T] = 0, \forall t \geq s \geq 0$ holds, which satisfies the second condition of Definition 3. ③ Combining conditions (20), (34) and using the same approach as shown in the proof of Lemma 3, we get $d_t[y_c(t), y_c(s)^T] = 0, d_t[y_c(s), y_c(t)^T] = 0$ and $d[y_c(t), y_c(t)^T] = 0$, for all $t \geq s \geq 0$ (here the symbol d_t denotes the forward differential with respect to t), which imply that $[y_c(t), y_c(s)^T] = 0$ holds for all $t, s \geq 0$ under the fact that $[y_c(0), y_c(0)^T] = 0$ given in Definition 3. Applying a similar trick, we have $[x_c(t), x_c(s)^T] = 0, [y_c(t), x_c(s)^T] = 0$ for all $t, s \geq 0$. We infer that output (16) and x_c are both classical in the sense of the third item of Definition 3. Therefore, we conclude that the system (12) is physically realizable in the sense of Definition 3, which shows that (33)-(35) are sufficient for realizability.

Conversely, now suppose that a system of the form (12) is physically realizable. It follows from Theorem 1 and the first item of Definition 3 that condition (37) holds. Then, reading off the first n rows and columns of both sides of (37) gives us condition (33). By the second item of Definition 3, we have condition (34) in the sense of Lemma 4. Since the system (12) is a *standard* form, it follows from the third item of Definition 1 that condition (35) holds. Therefore, constraints (33)-(35) are necessary for realizability. ■

B. Physical Realizability for the General Form

Without loss of generality, we need to give the physical realizability definition and constraints for the *general* form (11).

Definition 4: A system of the *general* form (11) is said to be physically realizable if its corresponding *standard* form (12) is physically realizable in the sense of Definition 3.

Theorem 4: A system of the *general* form (11) is physically realizable, if and only if the following constraints are satisfied:

$$\mathbf{A}\Theta_n + \Theta_n\mathbf{A}^T + \mathbf{B}\Theta_v\mathbf{B}^T = 0, \quad (44)$$

$$\mathbf{B}\Theta_v\mathbf{D}^T = -\Theta_n\mathbf{C}^T, \quad (45)$$

$$\mathbf{D}\Theta_v\mathbf{D}^T = \Theta_y. \quad (46)$$

Proof: Suppose that equations (44)-(46) hold. It follows from Theorem 2 that the general system (11) can be transformed to its corresponding *standard* system (12). Using relations (28) and equations (44)-(46), we get constraints (33)-(35). The corresponding *standard* system (12) is physically realizable in the sense of Theorem 3. Therefore, we conclude that (44)-(46) are sufficient for physical realizability.

Conversely, suppose that a system of the general form (11) is physically realizable. It follows from Definition 4 and Theorem 3 that constraints (33)-(35) hold. Conditions (44)-(46) can be obtained from constraints (33)-(35) by direct substitution using relations (28). Thus, constraints (44)-(46) are necessary for realizability. ■

V. SYSTEMATIC SYNTHESIS OF MIXED QUANTUM-CLASSICAL LINEAR STOCHASTIC SYSTEMS

By Theorem 2 and Definition 4, we know that a system of the general form (11) can be physically realized, if its corresponding *standard* form (12) is physically realizable. Therefore,

our purpose in this section is to develop a network synthesis theory only for a mixed quantum-classical system of the *standard* form (12) that generalizes the results in [21].

Lemma 5: The mixed quantum-classical linear stochastic system (12) is physically realizable if and only if conditions (18), (19), (20) and the constraints below are all satisfied

$$A_{qq}\Theta_{n_q} + \Theta_{n_q}A_{qq}^T + B_q\Theta_w B_q^T = 0, \quad (47)$$

$$A_{cq}\Theta_{n_q} + B_c\Theta_w B_q^T = 0, \quad (48)$$

$$B_c\Theta_w B_c^T = 0, \quad (49)$$

$$B_c\Theta_w D_q^T = 0, \quad (50)$$

$$B_q\Theta_w D_q^T = -\Theta_{n_q}C_{qq}^T, \quad (51)$$

$$B_c\Theta_w D_c^T = 0, \quad (52)$$

$$B_q\Theta_w D_c^T = -\Theta_{n_q}C_{cq}^T. \quad (53)$$

Proof: By Theorem 3, it is easily checked that conditions (18)-(20) are equivalent to (35) while (47)-(53) are equivalent to (33)-(34). ■

Lemma 6: If a matrix D_q satisfies the following condition

$$D_q\Theta_w D_q^T = \Theta_{y_q}, \quad (54)$$

then there exists a matrix N such that $\begin{bmatrix} D_q \\ N \end{bmatrix} \Theta_w \begin{bmatrix} D_q \\ N \end{bmatrix}^T = \Theta_w$, so that D_q can be embedded into a symplectic matrix $\tilde{V} = \begin{bmatrix} D_q^T & N^T \end{bmatrix}^T$.

Proof: The matrix D_q can be written in the form of

$$D_q = \begin{bmatrix} I & 0_{2n_{y_q} \times (2m-2n_{y_q})} \end{bmatrix} \begin{bmatrix} D_q \\ N \end{bmatrix}, \quad (55)$$

where N is a $(2m - 2n_{y_q}) \times 2m$ matrix. Let the rows of D_q be denoted by $d_1, d_2, \dots, d_{2n_{y_q}}$. Let $P(a|b_1, b_2, \dots, b_k)$ denote the orthogonal projection of the row vector a onto the subspace spanned by the row vectors b_1, b_2, \dots, b_k .

Now, we want to build a $(2m - 2n_{y_q}) \times 2m$ matrix N , following analogously the construction of the matrix V defined in [21, Lemma 6]. First, choose a row vector $v_1^{(1)} \in \mathbb{R}^{2m}$

linearly independent of $d_1, d_2, \dots, d_{2n_{yq}}$, and set $v_1^{(2)} = v_1^{(1)} - P(v_1^{(1)}|d_1, d_2, \dots, d_{2n_{yq}})$ and $v_1 = v_1^{(2)}\Theta_w$. Next, choose a row vector $v_2^{(1)} \in \mathbb{R}^{2m}$ linearly independent of $d_1, d_2, \dots, d_{2n_{yq}}$ and set $v_2^{(2)} = v_2^{(1)} - P(v_2^{(1)}|d_1, d_2, \dots, d_{2n_{yq}})$ and $v_2 = v_2^{(2)}\Theta_w$. Repeat this procedure analogously for $k = 3, \dots, m - n_{yq}$ to obtain vectors $v_3, v_4, \dots, v_{m-n_{yq}}$. Then, we choose a row vector $w_1^{(1)} \in \mathbb{R}^{2m}$ that is linearly independent of $d_1, d_2, \dots, d_{2n_{yq}}$ and $v_2, v_3, \dots, v_{m-n_{yq}}$ such that $(w_1^{(1)} - P(w_1^{(1)}|d_1, d_2, \dots, d_{2n_{yq}}, v_2, v_3, \dots, v_{m-n_{yq}}))v_1^T \neq 0$. Set $w_1^{(2)} = w_1^{(1)} - P(w_1^{(1)}|d_1, d_2, \dots, d_{2n_{yq}}, v_2, v_3, \dots, v_{m-n_{yq}})$ and $w_1 = w_1^{(2)}\Theta_w/(v_1w_1^{(2)T})$. Next, we choose $w_2^{(1)} \in \mathbb{R}^{2m}$ that is linearly independent of $d_1, d_2, \dots, d_{2n_{yq}}$ and $v_1, w_1, v_3, v_4, \dots, v_{m-n_{yq}}$ such that $(w_2^{(1)} - P(w_2^{(1)}|d_1, d_2, \dots, d_{2n_{yq}}, v_1, w_1, v_3, v_4, \dots, v_{m-n_{yq}}))v_2^T \neq 0$. Set $w_2^{(2)} = w_2^{(1)} - P(w_2^{(1)}|d_1, d_2, \dots, d_{2n_{yq}}, v_1, w_1, v_3, v_4, \dots, v_{m-n_{yq}})$ and $w_2 = w_2^{(2)}\Theta_w/(v_2w_2^{(2)T})$. Repeat the procedure in an analogous manner to construct $w_3, w_4, \dots, w_{m-n_{yq}}$. Then the matrix N is defined as

$$N = [v_1^T, w_1^T, v_2^T, w_2^T, \dots, v_{m-n_{yq}}^T, w_{m-n_{yq}}^T]^T. \quad (56)$$

By the above construction, we readily verify that the $2m \times 2m$ matrix $\tilde{V} = \begin{bmatrix} D_q^T & N^T \end{bmatrix}^T$ is a symplectic matrix ($\tilde{V}\Theta_w\tilde{V}^T = \Theta_w$) using equations (54) and (56). ■

Suppose that the system (12) is physically realizable. We are now in a position to explain how to realize the system (12) as an interconnection of a quantum system G_1 and a classical system G_2 . We define G_1 to be a fully quantum system given by

$$dx_q(t) = A_{qq}x_q(t)dt + B_qdw'(t) + Eu(t)dt, \quad (57)$$

$$dy_q(t) = C_{qq}x_q(t)dt + D_qdw'(t), \quad (58)$$

$$dy'_q(t) = C'_{qq}x_q(t)dt + D'_qdw'(t), \quad (59)$$

where $x_q, y_q, A_{qq}, B_q, C_{qq}, D_q$ are defined as in (13) and (15). Here $D'_q = N$ and $C'_{qq} = D'_q\Theta_wB_q^T\Theta_{n_q}$, where N can be obtained from D_q using Lemma 6. Note in particular that $D'_q\Theta_w(D'_q)^T = \Theta_{y'_q}$ and $\begin{bmatrix} D_q \\ D'_q \end{bmatrix} \Theta_w \begin{bmatrix} D_q \\ D'_q \end{bmatrix}^\top = \Theta_w$. Here $w'(t) = \begin{bmatrix} w'_1(t) \\ w'_2(t) \end{bmatrix}$, where $w'_1(t)$ and $w'_2(t)$ are two vectors of independent vacuum boson fields and will be defined later. The Hamiltonian of G_1 is given by $H_q = \frac{1}{2}x_q^TR_qx_q + x_q^TK_qu(t)$ with a real matrix $K_q = -\Theta_{n_q}E$; $u(t)$ a vector of real locally square integrable functions, representing a classical control signal; see [29], [30] for how to implement the linear part $x_q^TK_qu$ using classical devices. We then

define G_2 be a classical system given by

$$dx_c(t) = A'_{cc}x_c(t)dt + B'_c du_c(t), \quad (60)$$

$$dy_c(t) = C'_{cc}x_c(t)dt + D'_c du_c(t), \quad (61)$$

$$y'_{c1}(t) = C'_{c1}x_c(t), \quad y'_{c2}(t) = C'_{c2}x_c(t), \quad (62)$$

where x_c and y_c are defined as in (14) and (16). Here $u_c(t)$ is real locally square-integrable classical signal satisfying $du_c(t) = \alpha_c(t)dt + dw_c(t)$, where $w_c(t)$ is a vector of independent standard classical Wiener processes, and $\alpha_c(t)$ is a vector of real stochastic processes of locally bounded variation.

The remaining undefined system matrices, input and output signals appearing in (57)-(62) can be found in the following theorem, which presents a feedback architecture for the realization of the system (12).

Theorem 5: Assume that the system (12) is physically realizable and its system matrices are all already known. Let $C'_c = \begin{bmatrix} C'_{c1} \\ C'_{c2} \end{bmatrix}$ and there exist matrices C'_c , G , B'_c , and D'_c , such that

$$D_q C'_c = C_{qc}, \quad (63)$$

$$B'_c G C'_{qq} = A_{cq}, \quad (64)$$

$$B'_c G D'_q = B_c, \quad (65)$$

$$D'_c G C'_{qq} = C_{cq}, \quad (66)$$

$$D'_c G D'_q = D_c. \quad (67)$$

Then the feedback network shown in Figure 2, with the identification $u(t) = x_c(t)$, $du_c(t) = G dy'_q(t)$, $w'_1(t) = y'_{c1}(0) + \int_0^t y'_{c1}(s)ds + w_1(t)$, $w'_2(t) = y'_{c2}(0) + \int_0^t y'_{c2}(s)ds + w_2(t)$, $E = A_{qc} - B_q C'_c$, $A'_{cc} = A_{cc} - B'_c G D'_q C'_c$ and $C'_{cc} = C_{cc} - D'_c G D'_q C'_c$, is a physical realization of the system (12) consisting of a quantum system G_1 described by (57)-(59) and a classical system G_2 described by (60)-(62), where the network G can realize the matrix $G = KV$ to produce classical signals $u_c = G y'_q(t)$ satisfying $[u_c(t), u_c(s)^T] = 0, \forall t, s \geq 0$, where $K = [k_1^T, k_2^T, \dots, k_r^T]^T$ ($k_j = [0, 0, \dots, 0, 1, 0, \dots, 0] \in \mathbb{R}^{1 \times (n_c + n_{y_c})}$ with the 1 in the $(2j-1)$ -th position) and V is a symplectic matrix; the network S realizes a symplectic transformation $\begin{bmatrix} D_q \\ D'_q \end{bmatrix}$.

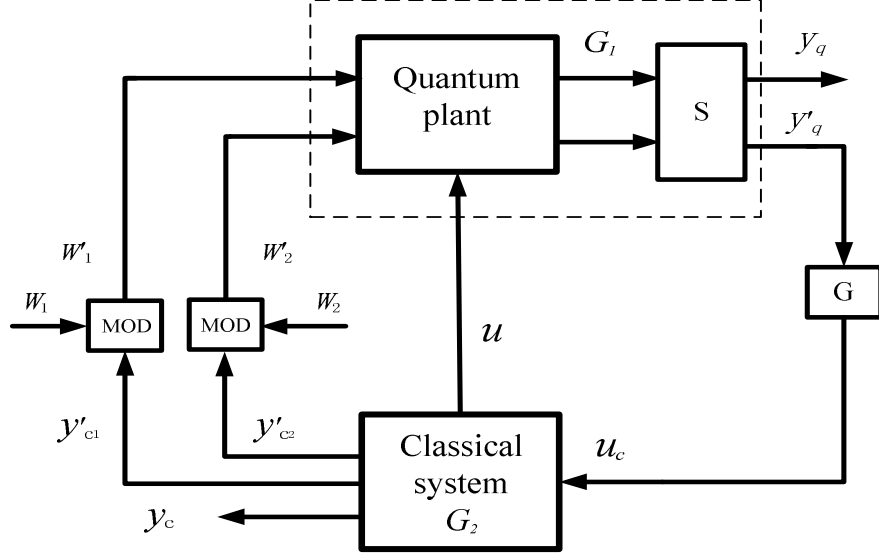


Fig. 2. Feedback interconnection of a quantum system G_1 and a classical system G_2 . The two sets of modulators (MODs) displace the vectors of vacuum quantum fields w_1 and w_2 to produce the quantum signals $w'_1(t)$ and $w'_2(t)$ by the classical vector signals $y'_{c1}(t)$ and $y'_{c2}(t)$, respectively. The network G corresponds to measurement processes.

Proof. First, we will show that under physical realization constraints (18)-(20) and (47)-(53), we can build matrices C'_c , G , B'_c , and D'_c to satisfy (63)-(67). It follows from equations (18) with invertible Θ_{y_q} that the wide matrix D_q has full row rank and $\text{rank}(D_q) = \text{rank} \left(\begin{bmatrix} D_q & C_{qc} \end{bmatrix} \right)$. So, the solution of equation (63) is written as $C'_c = D_q^T (D_q D_q^T)^{-1} C_{qc} + N(D_q)$, where $N(D_q)$ denotes a matrix of the same dimension as C'_c whose columns are in the kernel space of D_q . Let $\bar{B}_c = B'_c G$ and $\bar{D}_c = D'_c G$. Now we will show that the equation $\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix} D'_q = \begin{bmatrix} B_c \\ D_c \end{bmatrix}$ has solutions for $\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix}$. Combining equations (19) and (50) gives

$$\begin{bmatrix} D_q \\ D'_q \end{bmatrix} \Theta_w (D'_q)^T = \begin{bmatrix} 0_{2n_{yq} \times (2m-2n_{yq})} \\ \Theta_{y'_q} \end{bmatrix}, \quad (68)$$

$$\begin{bmatrix} D_q \\ D'_q \end{bmatrix} \Theta_w \begin{bmatrix} B_c^T & D_c^T \end{bmatrix} = \begin{bmatrix} 0_{2n_{yq} \times (n_c+n_{yc})} \\ D'_q \Theta_w \begin{bmatrix} B_c^T & D_c^T \end{bmatrix} \end{bmatrix}, \quad (69)$$

where $\Theta_{y'_q} = \text{diag}_{(m-n_{yq})}(J)$. From equations (68) and (69), we can infer that $\text{rank}((D'_q)^T) = \text{rank}(\Theta_{y'_q})$, $\text{rank} \left(\begin{bmatrix} B_c^T & D_c^T \end{bmatrix} \right) = \text{rank} \left(D'_q \Theta_w \begin{bmatrix} B_c^T & D_c^T \end{bmatrix} \right)$. Given that $\Theta_{y'_q}$ has full row rank, we

can conclude that $\text{rank}(\Theta_{y'_q}) = \text{rank}\left(\begin{bmatrix} \Theta_{y'_q} & D'_q \Theta_w \begin{bmatrix} B_c^T & D_c^T \end{bmatrix} \end{bmatrix}\right)$, which implies that $\text{rank}((D'_q)^T) = \text{rank}\left(\begin{bmatrix} (D'_q)^T & B_c^T & D_c^T \end{bmatrix}\right)$. So, there exist \bar{B}_c and \bar{D}_c satisfying (65) and (67), respectively. From constraints (48), (53), and $C'_{qq} = D'_q \Theta_w B_q^T \Theta_{n_q}$ defined as before, we conclude that equation (65) implies (64), and (67) implies (66), respectively.

Then it is straightforward to verify from (63)-(67) that interconnecting the system G_1 and the system G_2 gives the *standard* form (12), or equivalently described by (13)-(16). Now let us check that the system G_1 is a physically realizable fully quantum system. It follows from conditions (18) and (47) that the system G_1 satisfies constraints (33) and (35) in the sense of Theorem 3 with matrices A , B , D , Θ_n and $\text{diag}(\Theta_{y_q}, 0_{n_{y_c} \times n_{y_c}})$ replaced by corresponding matrices A_{qq} , B_q , D_q , Θ_{n_q} and Θ_{y_q} , respectively. The system G_1 also satisfies constraint (34) with its matrices replaced by corresponding matrices in equations (60)-(62) with the proof as follows:

$$\begin{aligned} -\Theta_{n_q} \left(\begin{bmatrix} D_q \\ D'_q \end{bmatrix} \Theta_w B_q^T \Theta_{n_q} \right)^T &= -\Theta_{n_q} \Theta_{n_q}^T B_q \Theta_w^T \begin{bmatrix} D_q \\ D'_q \end{bmatrix}^T \\ &= B_q \Theta_w \begin{bmatrix} D_q \\ D'_q \end{bmatrix}^T. \end{aligned}$$

So, the system G_1 is a physically realizable quantum system, where y'_q is the input to the network G . Given that $D'_q \Theta_w (D'_q)^T = \Theta_{y'_q} = \text{diag}_{(m-n_{y_q})}(J)$, we get from equations (20), (49), (52), (65) and (67) that

$$\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix} \Theta_{y'_q} \begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix}^T = 0. \quad (70)$$

From equation (70), we know that the matrix $\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix}$ with $\text{rank}\left(\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix}\right) = r$ can be decomposed as $\begin{bmatrix} \bar{B}_c \\ \bar{D}_c \end{bmatrix} = PZKV = \begin{bmatrix} P_1 Z \\ P_2 Z \end{bmatrix} KV$ (see [21, Lemma 6] for details), where $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ is a permutation matrix; Z is a matrix of the form $Z = \begin{bmatrix} I_r \\ X \end{bmatrix}$ if $r < n_c + n_{y_c}$, where X is some $(n_c + n_{y_c} - r) \times r$ matrix, $Z = I_{(n_c + n_{y_c})}$ if $r = n_c + n_{y_c}$, and V is a symplectic matrix. So, we can define

$$G = KV, \quad B'_c = P_1 Z, \quad D'_c = P_2 Z,$$

and the symplectic transformation V can be realized as a suitable static quantum optical network [31]. Applying K to $Vy'_q(t)$ is to measure the first r amplitude quadrature components of $Vy'_q(t)$ to obtain the measurement result $u_c(t) = KVy'_q(t)$. So, G represents measurement processes [21]. Then we can show that

$$[u_c(t), u_c(s)^T] = G[y'_q(t), y'_q(s)^T]G^T = \delta_{ts}G\text{diag}_{n_{y'_q}/2}(J)G^T = \delta_{ts} \times 0 = 0, \forall t, s \geq 0,$$

which implies that u_c is a classical signal. Thus G_2 described by (60)-(62) is a classical system, where the classical vector signals $y'_{c_1}(t)$ and $y'_{c_2}(t)$ are used to produce the quantum signals $w'_1(t)$ and $w'_2(t)$ which are then injected into G_1 . ■

Now an example is given to check our main results.

Example 2: Consider a mixed quantum-classical system of the *standard* form with A, B, C, D satisfying conditions (33)-(35), such that

$$A = \begin{bmatrix} -9 & -3 & -1 \\ 1 & -7 & -3 \\ -0.72 & -0.6 & -12 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -7 & 0 & -3 & 5 \\ 2 & 5 & 1 & -3 & 6 & -8 \\ 0 & 0.12 & 0 & 0 & 0 & -0.16 \end{bmatrix},$$

$$C = \begin{bmatrix} 38 & 46 & -42 \\ 0.31 & 0.4 & 0.35 \\ 4.2 & -6 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 & 10 & 0 & 6 & 0 \\ 0 & 0.04 & 0 & 0.05 & 0 & 0.03 \\ 0 & 0.8 & 0 & -1 & 0 & 0.6 \end{bmatrix}.$$

Following the construction in the proof of Theorem 5, we have the quantum system G_1 given by

$$dx_q(t) = \begin{bmatrix} -9 & -3 \\ 1 & -7 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 1 & 2 & -7 & 0 & -3 & 5 \\ 2 & 5 & 1 & -3 & 6 & -8 \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix} + \begin{bmatrix} -30.4 \\ 22.2 \end{bmatrix} du(t),$$

$$dy_q(t) = \begin{bmatrix} 38 & 46 \\ 0.31 & 0.4 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 8 & 0 & 10 & 0 & 6 & 0 \\ 0 & 0.04 & 0 & 0.05 & 0 & 0.03 \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix},$$

$$dy'_q(t) = \begin{bmatrix} dy'_{q_1}(t) \\ dy'_{q_2}(t) \end{bmatrix} = \begin{bmatrix} -1.1 & 2.3 \\ 4.2 & -6 \\ -47 & -14 \\ -0.72 & -0.6 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 0.4 & 0 & -0.5 & 0 & 0.3 & 0 \\ 0 & 0.8 & 0 & -1 & 0 & 0.6 \\ 3 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0.12 & 0 & 0 & 0 & -0.16 \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix} \times$$

and the classical system G_2 described by

$$\begin{aligned} dx_c(t) &= -12x_c(t) + [3.6836 \quad -0.4345]du_c(t), \\ dy_c(t) &= 12x_c(t)dt + [-0.2065 \quad 1.2388]du_c(t), \\ y'_{c1}(t) &= 0, \\ y'_{c2}(t) &= \begin{bmatrix} y'_{c21}(t) \\ y'_{c22}(t) \end{bmatrix} = \begin{bmatrix} -4.2 & 7 & 0 & 0 \end{bmatrix}^T x_c(t). \end{aligned}$$

It can be easily checked that the closed loop system described by (12) with the above matrices A, B, C, D is obtained by making the identification

$$\begin{aligned} u(t) &= x_c(t), \\ du_c(t) &= \begin{bmatrix} 0.2086 & -0.7489 \\ 3.4253 & -4.9684 \end{bmatrix} x_q(t)dt + \begin{bmatrix} 0 & 0.1109 & 0 & -0.0971 & 0 & 0.014 \\ 0 & 0.6643 & 0 & -0.8235 & 0 & 0.4867 \end{bmatrix} \begin{bmatrix} dw_1(t) \\ dw_2(t) \end{bmatrix}, \\ dw'_1(t) &= dw_1(t), \\ dw'_2(t) &= \begin{bmatrix} dw'_{21}(t) \\ dw'_{22}(t) \end{bmatrix} = \begin{bmatrix} -4.2 & 7 & 0 & 0 \end{bmatrix}^T x_c(t) + \begin{bmatrix} dw_{21}(t) \\ dw_{22}(t) \end{bmatrix}, \\ \text{where } du_c(t) &= \begin{bmatrix} du_{c1} \\ du_{c2} \end{bmatrix} \text{ and } dw_2 = \begin{bmatrix} dw_{21}(t) \\ dw_{22}(t) \end{bmatrix}; \text{ the matrix } G = \begin{bmatrix} 0 & 0.0971 & 0 & 0.2769 \\ 0 & 0.8235 & 0 & 0.0462 \end{bmatrix}. \end{aligned}$$

The realization of this mixed system is shown in Figure 3. The details of the construction and the individual components involved can be found in [4], [11],[20] and the references therein.

VI. CONCLUSION

In this paper, two forms (a general form and a *standard* form) are presented for the physical realization of the mixed quantum-classical linear stochastic system. We have shown the relation between these two forms. Three physical realization constraints are derived for the *standard* form and the *general* form, respectively. A network theory is developed for synthesizing linear dynamical mixed quantum-classical stochastic systems of the *standard* form in a systematic way. One feedback architecture is proposed for this realization.

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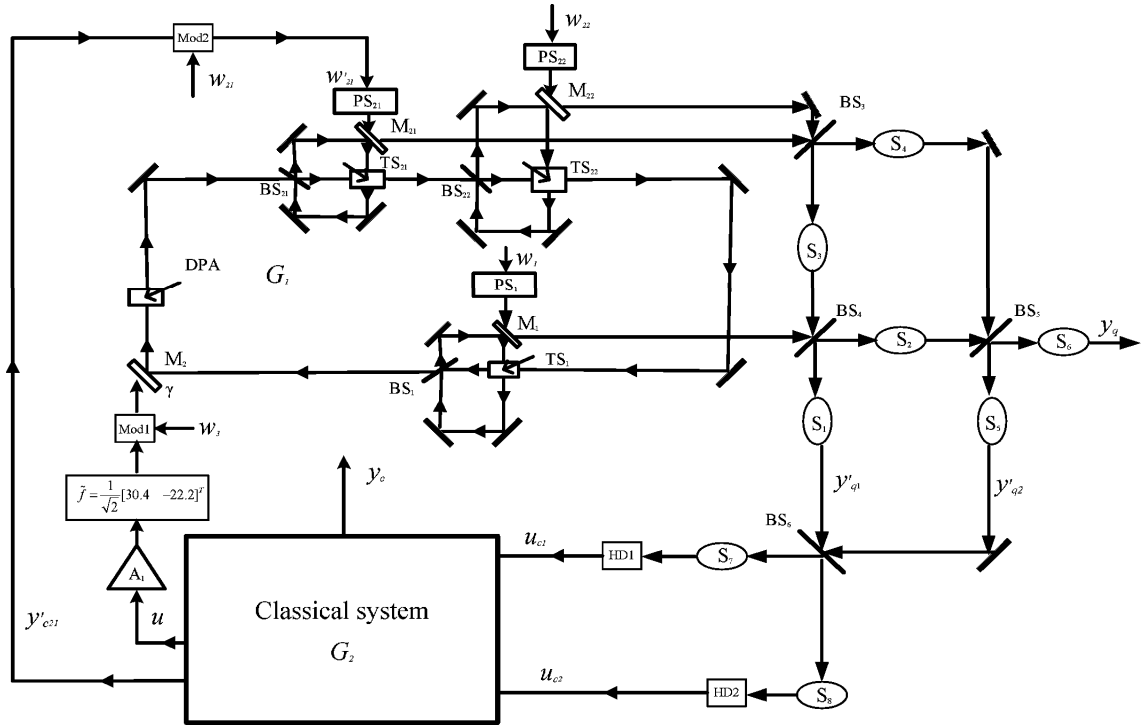


Fig. 3. A realization of the mixed quantum-classical system. Black rectangles denote fully reflecting mirrors. M_1, M_{21}, M_{22} and M_3 represent transmitting mirrors with coupling constants $\kappa_1, \kappa_{21}, \kappa_{22}$ and γ , respectively ($\gamma \ll 1, \gamma \ll \kappa_1, \kappa_{21}, \kappa_{22}$); $BS_1, BS_{21}, BS_{22}, BS_3, BS_4, BS_5$ and BS_6 represent beam splitters; TS_1, TS_{21} and TS_{22} represent two-mode squeezers; PS_1, PS_{21}, PS_{22} represent phase shifters; S_i ($i = 1, 2, \dots, 8$) represents a squeezer; DPA is short for degenerate parametric amplifier; Mod_i ($i = 1, 2, 3, 4$) represents a modulator; HD_i ($i = 1, 2$) represents a homodyne detector; A_1 is a amplifier with gain $\frac{1}{\sqrt{\gamma}}$. \tilde{f} can be realized using a computer. w_1, w_{21}, w_{22}, w_3 are vacuum noises and the contribution of w_3 to quantum system noise is negligible compared to that of other vacuum noises.

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